Analytic expressions of potential operators in circular and spherical geometries

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This document aims at giving explicit expressions for layer potentials for various classical equations in circular and spherical geometries. These expressions will then be used to implement reference solutions in the boundary element library `BemTool`. More precisely, these reference solutions will be stored under the form of routines located in the folder `bemtool/miscellaneous/` of the library.

For each considered case, we specify a domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3, and denote $\Gamma = \partial\Omega$ its boundary. We denote $\gamma_{\mathrm{D}} : \mathrm{H}^1(\Omega) \to \mathrm{H}^{1/2}(\Gamma)$ the interior Dirichlet trace defined by $\gamma_{\mathrm{D}}(u) := u|_\Gamma$ for any $u \in \mathscr{C}^0(\overline{\Omega})$, and $\gamma_{\mathrm{N}} : \mathrm{H}^1(\Delta, \Omega) \to \mathrm{H}^{-1/2}(\Gamma)$ the interior Neumann trace defined by $\gamma_{\mathrm{N}}(u) := \boldsymbol{n} \cdot \nabla u|_\Gamma = \partial_r u|_\Gamma$ where $\boldsymbol{n}$ refers to the normal vector field directed toward the exterior of Ω. We define $\gamma_{\mathrm{D},c}, \gamma_{\mathrm{N},c}$ in the same manner, except that the traces are taken from the exterior of Ω. Finally, we set

$$\{\gamma_{\mathrm{D}}\} := (\gamma_{\mathrm{D}} + \gamma_{\mathrm{D},c})/2 \quad \{\gamma_{\mathrm{N}}\} := (\gamma_{\mathrm{N}} + \gamma_{\mathrm{N},c})/2$$

$$[\gamma_{\mathrm{D}}] := \gamma_{\mathrm{D}} - \gamma_{\mathrm{D},c} \qquad [\gamma_{\mathrm{N}}] := \gamma_{\mathrm{N}} - \gamma_{\mathrm{N},c}.$$

Let us introduce the layer potentials associated to the interior of Ω. For any trace $v \in \mathrm{H}^{+1/2}(\Gamma), p \in \mathrm{H}^{-1/2}(\Gamma)$, their explicit expression is given by:

$$\begin{aligned}
\mathrm{SL}(p)(\boldsymbol{x}) &:= \int_\Gamma \mathscr{G}(\boldsymbol{x} - \boldsymbol{y}) p(\boldsymbol{y}) d\sigma(\boldsymbol{y}), \\
\mathrm{DL}(p)(\boldsymbol{x}) &:= \int_\Gamma \boldsymbol{n}(\boldsymbol{y}) \cdot (\nabla\mathscr{G})(\boldsymbol{x} - \boldsymbol{y}) p(\boldsymbol{y}) d\sigma(\boldsymbol{y}).
\end{aligned} \tag{1}$$

1   Analytic solutions in 2-D

In this section $\Omega = \mathrm{D} \subset \mathbb{R}^2$ is the disc of center 0 and radius $r_\star > 0$. For a given point $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, we shall write (r, θ) to refer to its polar coordinates centered at 0,

$$\begin{cases} x_1 = r\cos\theta, \\ x_2 = r\sin\theta. \end{cases}$$

We shall use Fourier harmonics $\mathfrak{e}_n(\theta) := \exp(\imath n\theta)$ and write $\mathfrak{e}_n(\boldsymbol{x}/|\boldsymbol{x}|) := \mathfrak{e}_n(\theta)$. We will only consider equation that admit rotational symetry (Laplace and Helmholtz equations), which will at the end of the day implies

$$\int_\Gamma \bar{\mathfrak{e}}_n \cdot \gamma_* \mathrm{DL}(\mathfrak{e}_p) d\sigma = \int_\Gamma \bar{\mathfrak{e}}_n \cdot \gamma_* \mathrm{SL}(\mathfrak{e}_p) d\sigma = 0$$
$$\text{for } * = \mathrm{D}, \mathrm{N}, \quad \text{and} \quad n \neq p.$$

We shall then deliver explicit expressions for the coefficients $\int_\Gamma \bar{\mathfrak{e}}_n \cdot \{\gamma_*\}\mathrm{DL}(\mathfrak{e}_n)d\sigma$ and $\int_\Gamma \bar{\mathfrak{e}}_n \cdot \{\gamma_*\}\mathrm{SL}(\mathfrak{e}_n)d\sigma$, for arbitrary $n \in \mathbb{Z}$, and for $* = \mathrm{D}, \mathrm{N}$.

1.1 Laplace equation

In this paragraph, we first consider the Laplace equation $\Delta u = 0$ in $\mathbb{R}^2 \backslash \Gamma$ with decay condition at infinity. The Green kernel of this equation is $\mathscr{G}(\boldsymbol{x}) := -(2\pi)^{-1} \ln|\boldsymbol{x}|$. Note that, for Laplace equation in 2D, the behaviour of layer potentials associated to the Fourier harmonics $\mathfrak{e}_n$ with $n = 0$ is somewhat special. Let us leave aside this case ate first and consider $n \in \mathbb{Z} \backslash \{0\}$. The layer potentials are then given by

$$\mathrm{SL}(\mathfrak{e}_n)(\boldsymbol{x}) = \begin{cases} \dfrac{r_\star}{2|n|}\left(\dfrac{|\boldsymbol{x}|}{r_\star}\right)^{-|n|} \mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > r_\star \\[2ex] \dfrac{r_\star}{2|n|}\left(\dfrac{|\boldsymbol{x}|}{r_\star}\right)^{+|n|} \mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < r_\star \end{cases}$$

and

$$\mathrm{DL}(\mathfrak{e}_n)(\boldsymbol{x}) = \begin{cases} -\dfrac{1}{2}\left(\dfrac{|\boldsymbol{x}|}{r_\star}\right)^{-|n|} \mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > r_\star \\[2ex] +\dfrac{1}{2}\left(\dfrac{|\boldsymbol{x}|}{r_\star}\right)^{+|n|} \mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < r_\star \end{cases}$$

The boundary integral operators are given by

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_\mathrm{D}\}\mathrm{SL}(\mathfrak{e}_n)d\sigma = \pi r_\star^2/|n|,$$

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_\mathrm{N}\}\mathrm{DL}(\mathfrak{e}_n)d\sigma = \pi|n|,$$

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_\mathrm{D}\}\mathrm{DL}(\mathfrak{e}_n)d\sigma = 0,$$

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_\mathrm{N}\}\mathrm{SL}(\mathfrak{e}_n)d\sigma = 0.$$

For the case $n = 0$ we have $\mathfrak{e}_0(\boldsymbol{x}) = 1$. The layer potentials are given by

$$\mathrm{SL}(\mathfrak{e}_0) = \begin{cases} -r_\star \ln(|\boldsymbol{x}|) & \text{for} \quad |\boldsymbol{x}| > r_\star \\ -r_\star \ln(r_\star) & \text{for} \quad |\boldsymbol{x}| < r_\star \end{cases}$$

$$\mathrm{DL}(\mathfrak{e}_0) = 1_\mathrm{D}(\boldsymbol{x})$$

and the expression of the corresponding integral operators is easily deduced

$$\int_\Gamma \bar{\mathfrak{e}}_0 \{\gamma_\mathrm{D}\}\mathrm{SL}(\mathfrak{e}_0)d\sigma = -2\pi r_\star^2 \ln(r_\star),$$

$$\int_\Gamma \bar{\mathfrak{e}}_0 \{\gamma_\mathrm{N}\}\mathrm{DL}(\mathfrak{e}_0)d\sigma = 0,$$

$$\int_\Gamma \bar{\mathfrak{e}}_0 \{\gamma_\mathrm{D}\}\mathrm{DL}(\mathfrak{e}_0)d\sigma = +\pi r_\star,$$

$$\int_\Gamma \bar{\mathfrak{e}}_0 \{\gamma_\mathrm{N}\}\mathrm{SL}(\mathfrak{e}_0)d\sigma = -\pi r_\star.$$

1.2 Helmholtz equation

Here we consider an Helmholtz equation $-\Delta u - \kappa^2 u = 0$ with outgoing radiation condition. The corresponding Green kernel is $\mathscr{G}(\boldsymbol{x}) = \frac{\imath}{4}H_0^{(1)}(\kappa|\boldsymbol{x}|)$, where $H_0^{(1)}(z)$ refers to the Hankel function of order zero and of the first kind, see §10.2 and Formula 10.4.3 in [1]. The layer potentials admit the expressions

$$
\mathrm{SL}(\mathfrak{e}_n)(\boldsymbol{x}) = \begin{cases} \imath r_\star \dfrac{\pi}{2} J_{|n|}(\kappa r_\star) H_{|n|}^{(1)}(\kappa|\boldsymbol{x}|)\mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > r_\star \\[3mm] \imath r_\star \dfrac{\pi}{2} H_{|n|}^{(1)}(\kappa r_\star) J_{|n|}(\kappa|\boldsymbol{x}|)\mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < r_\star \end{cases}
$$

and

$$
\mathrm{DL}(\mathfrak{e}_n)(\boldsymbol{x}) = \begin{cases} -\imath\kappa r_\star \dfrac{\pi}{2} H_{|n|}^{(1)'}(\kappa r_\star) J_{|n|}(\kappa|\boldsymbol{x}|)\mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < r_\star \\[3mm] -\imath\kappa r_\star \dfrac{\pi}{2} J_{|n|}'(\kappa r_\star) H_{|n|}^{(1)}(\kappa|\boldsymbol{x}|)\mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > r_\star \end{cases}
$$

The boundary integral operators are given by

$$
\int_\Gamma \bar{\mathfrak{e}}_n\{\gamma_{\mathrm{D}}\}\mathrm{SL}(\mathfrak{e}_n)d\sigma = \imath r_\star^2 \pi^2 H_{|n|}^{(1)}(\kappa r_\star) J_{|n|}(\kappa r_\star)
$$

$$
\int_\Gamma \bar{\mathfrak{e}}_n\{\gamma_{\mathrm{N}}\}\mathrm{DL}(\mathfrak{e}_n)d\sigma = -\imath\kappa^2 r_\star^2 \pi^2 H_{|n|}^{(1)'}(\kappa r_\star) J_{|n|}'(\kappa r_\star)
$$

$$
\int_\Gamma \bar{\mathfrak{e}}_n\{\gamma_{\mathrm{N}}\}\mathrm{SL}(\mathfrak{e}_n)d\sigma = +r_\star^2 \imath\kappa\frac{\pi^2}{2}\left(H_{|n|}^{(1)}(\kappa r_\star) J_{|n|}'(\kappa r_\star) + H_{|n|}^{(1)'}(\kappa r_\star) J_{|n|}(\kappa r_\star) \right)
$$

$$
\int_\Gamma \bar{\mathfrak{e}}_n\{\gamma_{\mathrm{D}}\}\mathrm{DL}(\mathfrak{e}_n)d\sigma = -\imath\kappa r_\star^2 \frac{\pi^2}{2}\left(H_{|n|}^{(1)}(\kappa r_\star) J_{|n|}'(\kappa r_\star) + H_{|n|}^{(1)'}(\kappa r_\star) J_{|n|}(\kappa r_\star) \right)
$$

1.3 Modified Helmholtz equation

Here we consider a modified Helmholtz equation $-\Delta u + \kappa^2 u = 0$ with outgoing radiation condition. The corresponding Green kernel is $\mathscr{G}(\boldsymbol{x}) = \frac{1}{2\pi}K_0(\kappa|\boldsymbol{x}|)$, where $K_0(z)$ refers to the modified Bessel function of order zero and of the second kind, see §10.25 [1]. We also denote I_0, the modified Bessel function of order zero and of the first kind. The layer potentials admit the expressions

$$
\mathrm{SL}(\mathfrak{e}_n)(\boldsymbol{x}) = \begin{cases} r_\star K_{|n|}(\kappa|\boldsymbol{x}|) I_{|n|}(\kappa r_\star)\mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > r_\star \\[3mm] r_\star K_{|n|}(\kappa r_\star) I_{|n|}(\kappa|\boldsymbol{x}|)\mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < r_\star \end{cases}
$$

and

$$
\mathrm{DL}(\mathfrak{e}_n)(\boldsymbol{x}) = \begin{cases} -\kappa r_\star I_{|n|}'(\kappa r_\star) K_{|n|}(\kappa|\boldsymbol{x}|)\mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < r_\star \\[3mm] -\kappa r_\star K_{|n|}'(\kappa r_\star) I_{|n|}(\kappa|\boldsymbol{x}|)\mathfrak{e}_n\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > r_\star \end{cases}
$$

The boundary integral operators are given by

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_D\} SL(\mathfrak{e}_n) d\sigma = 2\pi r_\star^2 K_{|n|}(\kappa r_\star) I_{|n|}(\kappa r_\star)$$

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_N\} DL(\mathfrak{e}_n) d\sigma = -2\kappa^2 r_\star^2 \pi I'_{|n|}(\kappa r_\star) K'_{|n|}(\kappa r_\star)$$

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_N\} SL(\mathfrak{e}_n) d\sigma = +\pi r_\star^2 \kappa \big(K'_{|n|}(\kappa r_\star) I_{|n|}(\kappa r_\star) + K_{|n|}(\kappa r_\star) I'_{|n|}(\kappa r_\star) \big)$$

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_D\} DL(\mathfrak{e}_n) d\sigma = -\kappa r_\star^2 \pi \big(I'_{|n|}(\kappa r_\star) K_{|n|}(\kappa r_\star) + K'_{|n|}(\kappa r_\star) I_{|n|}(\kappa r_\star) \big)$$

Notice that we can check the previous relations formally, using the relation obtained for the Helmholtz equation with the transformation $\kappa \to \iota\kappa$, and the following relations for $x \in \mathbb{R}$ (see Formula 10.27.6 and 10.27.8 in [1]):

$$K_{|n|}(x) = \frac{\pi}{2} \iota^{|n|+1} H_{|n|}^{(1)}(\iota x)$$

$$I_{|n|}(x) = \iota^{-|n|} J_{|n|}(\iota x)$$

2 Analytic solutions in 3-D

In this section $\Omega = B \subset \mathbb{R}^3$ is the ball of center 0 and radius $\rho_\star > 0$. For a given point $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, we shall write $(\rho, \theta, \varphi) \in \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi[$ to refer to its polar coordinates centered at 0,

$$\begin{cases} x_1 = \rho \sin\theta \cos\varphi, \\ x_2 = \rho \sin\theta \sin\varphi, \\ x_3 = \rho \cos\theta. \end{cases}$$

To conduct separation of variables, we use the so-called spherical harmonics $Y_l^m(\theta, \varphi)$ and sometimes write $Y_l^m(\boldsymbol{x}/|\boldsymbol{x}|) := Y_l^m(\theta, \varphi)$. Here the indices l, m have to satisfy $0 \le |m| \le l$. For the definition of these functions, we use the convention of the `boost::math` library, namely

$$Y_l^m(\theta, \varphi) := \sqrt{\frac{(l+1/2)}{2\pi} \frac{(l-|m|)!}{(l+|m|)!}} \, P_l^m(\cos\theta) \exp(\iota m\varphi).$$

Here the $P_l^m(z)$ are the associated Legendre functions. With thsis definition, the family (Y_l^m) forms an orthonormal basis of $L^2(S^2)$ where S^2 is the unit sphere. Here again, we consider only rotation invariant equations (Laplace, Helmholtz and Maxwell), so that the layer potentials will be diagonalised by the spherical harmonics

$$\int_\Gamma \overline{Y}_l^m \cdot \gamma_* DL(Y_p^q) d\sigma = \int_\Gamma \overline{Y}_l^m \cdot \gamma_* SL(Y_p^q) d\sigma = 0$$

for $* = D, N, \quad$ and $(l, m) \ne (p, q)$.

2.1 Laplace equation

In this paragraph, we first consider the Laplace equation $\Delta u = 0$ in $\mathbb{R}^3 \backslash \Gamma$ with decay condition at infinity. The Green kernel of this equation is $\mathscr{G}(\boldsymbol{x}) := 1/(4\pi|\boldsymbol{x}|)$. The layer potentials are

given by

$$\mathrm{SL}(\mathrm{Y}_l^m)(\boldsymbol{x}) = \begin{cases} \dfrac{\rho_\star}{2l+1}\left(\dfrac{|\boldsymbol{x}|}{\rho_\star}\right)^{-(l+1)}\mathrm{Y}_l^m\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > \rho_\star \\[2ex] \dfrac{\rho_\star}{2l+1}\left(\dfrac{|\boldsymbol{x}|}{\rho_\star}\right)^{+l}\mathrm{Y}_l^m\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < \rho_\star \end{cases}$$

and

$$\mathrm{DL}(\mathrm{Y}_l^m)(\boldsymbol{x}) = \begin{cases} -\dfrac{l}{2l+1}\left(\dfrac{|\boldsymbol{x}|}{\rho_\star}\right)^{-(l+1)}\mathrm{Y}_l^m\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > \rho_\star \\[2ex] +\dfrac{l+1}{2l+1}\left(\dfrac{|\boldsymbol{x}|}{\rho_\star}\right)^{+l}\mathrm{Y}_l^m\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < \rho_\star \end{cases}$$

The boundary integral operators are given by

$$\int_\Gamma \overline{\mathrm{Y}}_l^m\{\gamma_{\mathrm{D}}\}\mathrm{SL}(\mathrm{Y}_l^m)d\sigma = \frac{\rho_\star^3}{2l+1}$$

$$\int_\Gamma \overline{\mathrm{Y}}_l^m\{\gamma_{\mathrm{N}}\}\mathrm{DL}(\mathrm{Y}_l^m)d\sigma = \frac{l(l+1)}{2l+1}\rho_\star$$

$$\int_\Gamma \overline{\mathrm{Y}}_l^m\{\gamma_{\mathrm{N}}\}\mathrm{SL}(\mathrm{Y}_l^m)d\sigma = -\frac{1}{2}\frac{\rho_\star^2}{2l+1}$$

$$\int_\Gamma \overline{\mathrm{Y}}_l^m\{\gamma_{\mathrm{D}}\}\mathrm{DL}(\mathrm{Y}_l^m)d\sigma = +\frac{1}{2}\frac{\rho_\star^2}{2l+1}$$

2.2 Helmholtz equation

Here we consider Helmholtz equation $-\Delta u - \kappa^2 u = 0$ with wave number $\kappa > 0$. The outgoing Green kernel is given here by $\mathscr{G}_\kappa(\boldsymbol{x}) := \exp(\imath\kappa|\boldsymbol{x}|)/(4\pi|\boldsymbol{x}|)$. Define the spherical Bessel functions $j_l(z) := \sqrt{\pi/(2z)}J_{l+1/2}(z)$ and the spherical Hankel functions $h_l^{(1)}(z) := \sqrt{\pi/(2z)}H_{l+1/2}^{(1)}(z)$ like in §10.47 of [1]. The single layer and double layer potentials admit explicit expressions in terms of the spherical harmonics. On the one hand

$$\mathrm{SL}(\mathrm{Y}_l^m)(\boldsymbol{x}) = \begin{cases} +\imath\kappa\rho_\star^2\, h_l^{(1)}(\kappa\rho_\star)j_l(\kappa|\boldsymbol{x}|)\mathrm{Y}_l^m\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < \rho_\star \\[2ex] +\imath\kappa\rho_\star^2\, j_l(\kappa\rho_\star)h_l^{(1)}(\kappa|\boldsymbol{x}|)\mathrm{Y}_l^m\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > \rho_\star. \end{cases}$$

and

$$\mathrm{DL}(\mathrm{Y}_l^m)(\boldsymbol{x}) = \begin{cases} -\imath\kappa^2\rho_\star^2\, j_l(\kappa|\boldsymbol{x}|)h_l^{(1)'}(\kappa\rho_\star)\mathrm{Y}_l^m\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| < \rho_\star \\[2ex] -\imath\kappa^2\rho_\star^2\, h_l^{(1)}(\kappa|\boldsymbol{x}|)j_l'(\kappa\rho_\star)\mathrm{Y}_l^m\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) & \text{for} \quad |\boldsymbol{x}| > \rho_\star. \end{cases}$$

The boundary integral operators are given by

$$\int_\Gamma \overline{Y}_l^m \{\gamma_\mathrm{D}\} \mathrm{SL}(Y_l^m) d\sigma = +\imath\kappa\rho_\star^4\, j_l(\kappa\rho_\star)h_l^{(1)}(\kappa\rho_\star),$$

$$\int_\Gamma \overline{Y}_l^m \{\gamma_\mathrm{N}\} \mathrm{DL}(Y_l^m) d\sigma = -\imath\kappa^3\rho_\star^4\, j_l'(\kappa\rho_\star)h_l^{(1)'}(\kappa\rho_\star),$$

$$\int_\Gamma \overline{Y}_l^m \{\gamma_\mathrm{N}\} \mathrm{SL}(Y_l^m) d\sigma = +\imath\frac{\kappa^2}{2}\rho_\star^4\big(j_l'(\kappa\rho_\star)h_l^{(1)}(\kappa\rho_\star) + j_l(\kappa\rho_\star)h_l^{(1)'}(\kappa\rho_\star)\big),$$

$$\int_\Gamma \overline{Y}_l^m \{\gamma_\mathrm{D}\} \mathrm{DL}(Y_l^m) d\sigma = -\imath\frac{\kappa^2}{2}\rho_\star^4\big(j_l'(\kappa\rho_\star)h_l^{(1)}(\kappa\rho_\star) + j_l(\kappa\rho_\star)h_l^{(1)'}(\kappa\rho_\star)\big).$$

2.3   Modified Helmholtz equation

Here we consider Helmholtz equation $-\Delta u + \kappa^2 u = 0$ with wave number $\kappa > 0$. The outgoing Green kernel is given here by $\mathscr{G}_\kappa(\boldsymbol{x}) := \exp(-\imath\kappa|\boldsymbol{x}|)/(4\pi|\boldsymbol{x}|)$. Define the modified spherical Bessel functions $i_l(z) := \sqrt{\pi/(2z)}I_{l+1/2}(z)$ and the modified spherical Hankel functions $k_l(z) := \sqrt{\pi/(2z)}K_{l+1/2}(z)$ like in §10.47 of [1]. The single layer and double layer potentials admit explicit expressions in terms of the spherical harmonics. On the one hand

$$\mathrm{SL}(Y_l^m)(\boldsymbol{x}) = \begin{cases} +\dfrac{2\kappa\rho_\star^2}{\pi} k_l(\kappa|\boldsymbol{x}|)i_l(\kappa\rho_\star)Y_l^m\Big(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\Big) & \text{for } |\boldsymbol{x}| < \rho_\star \\[2ex] +\dfrac{2\kappa\rho_\star^2}{\pi} k_l(\kappa\rho_\star)i_l(\kappa|\boldsymbol{x}|)Y_l^m\Big(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\Big) & \text{for } |\boldsymbol{x}| > \rho_\star. \end{cases}$$

and

$$\mathrm{DL}(Y_l^m)(\boldsymbol{x}) = \begin{cases} -\dfrac{2\kappa^2\rho_\star^2}{\pi} k_l(\kappa|\boldsymbol{x}|)i_l'(\kappa\rho_\star)Y_l^m\Big(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\Big) & \text{for } |\boldsymbol{x}| < \rho_\star \\[2ex] -\dfrac{2\kappa^2\rho_\star^2}{\pi} k_l'(\kappa\rho_\star)i_l(\kappa|\boldsymbol{x}|)Y_l^m\Big(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\Big) & \text{for } |\boldsymbol{x}| > \rho_\star. \end{cases}$$

The boundary integral operators are given by

$$\int_\Gamma \overline{Y}_l^m \{\gamma_\mathrm{D}\} \mathrm{SL}(Y_l^m) d\sigma = +\frac{2\kappa\rho_\star^4}{\pi} i_l(\kappa\rho_\star)k_l(\kappa\rho_\star),$$

$$\int_\Gamma \overline{Y}_l^m \{\gamma_\mathrm{N}\} \mathrm{DL}(Y_l^m) d\sigma = -\frac{2\kappa^3\rho_\star^4}{\pi} i_l'(\kappa\rho_\star)k_l'(\kappa\rho_\star),$$

$$\int_\Gamma \overline{Y}_l^m \{\gamma_\mathrm{N}\} \mathrm{SL}(Y_l^m) d\sigma = +\frac{\kappa^2\rho_\star^4}{\pi}\big(i_l'(\kappa\rho_\star)k_l(\kappa\rho_\star) + i_l(\kappa\rho_\star)k_l'(\kappa\rho_\star)\big),$$

$$\int_\Gamma \overline{Y}_l^m \{\gamma_\mathrm{D}\} \mathrm{DL}(Y_l^m) d\sigma = -\frac{\kappa^2\rho_\star^4}{\pi}\big(i_l'(\kappa\rho_\star)k_l(\kappa\rho_\star) + i_l(\kappa\rho_\star)k_l'(\kappa\rho_\star)\big).$$

Notice that we can check the previous relations formally, using the relation obtained for the Helmholtz equation with the transformation $\kappa \to \imath\kappa$, and the following relations for $x \in \mathbb{R}$ (see Formula 10.47.7 and 10.47.9 in [1] and formulas in section 1.3):

$$k_l(x) = -\frac{\pi}{2}\imath^l h_l^{(1)}(\imath x)$$

$$i_l(x) = \imath^{-l} j_l(\imath x)$$

3 Analytic solutions to Maxwell's equations

In the present section we use once again the notations of Section 2, in particular concerning spherical coordinates and spherical harmonics. We need to introduce vector counterpart to spherical harmonics. Considering the $Y_l^m = Y_l^m(\vartheta)$ as functions on the unit sphere $\vartheta \in S^2 \subset \mathbb{R}^3$, we set

$$\mathbf{X}_{l,m}^+ = \frac{1}{\sqrt{l(l+1)}}\nabla_{S^2}Y_l^m \quad,\text{and},\quad \mathbf{X}_{l,m}^- = \frac{1}{\sqrt{l(l+1)}}\boldsymbol{n}_{S^2} \times \nabla_{S^2}Y_l^m$$

where $\boldsymbol{n}_{S^2}$ refers to the unit normal vector on the unit sphere directed toward the exterior of the unit ball. The set $\{\mathbf{X}_{l,m}^\pm\}$ yields an orthonormal basis of the space os square integrable tangential vector fields over S^2.

We only yield explicit expressions for the operators Electric Field Integral Equation (EFIE) and Magnetic Field Integral Equation (MFIE). In a general geometrical setting, these are defined by the following variationnal forms

$$\langle \text{EFIE}_\kappa(\boldsymbol{u}), \boldsymbol{v}\rangle := \int_{\Gamma\times\Gamma} \mathscr{G}_\kappa(\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{y}) - \kappa^{-2}\text{div}_\Gamma\boldsymbol{u}(\boldsymbol{x})\text{div}_\Gamma\boldsymbol{v}(\boldsymbol{y}) \big)d\sigma(\boldsymbol{x},\boldsymbol{y})$$

$$\langle \text{MFIE}_\kappa(\boldsymbol{u}), \boldsymbol{v}\rangle := \int_{\Gamma\times\Gamma} (\nabla\mathscr{G}_\kappa)(\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{v}(\boldsymbol{y})\times\boldsymbol{u}(\boldsymbol{x}))\, d\sigma(\boldsymbol{x},\boldsymbol{y})$$

$$\mathscr{G}_\kappa(\boldsymbol{x}) := \exp(\imath\kappa|\boldsymbol{x}|)/(4\pi|\boldsymbol{x}|)$$

In the case $\Gamma = S^2$ we have:

$$\langle \text{EFIE}_\kappa(\mathbf{X}_{l,m}^+), \mathbf{X}_{l,m}^+\rangle = (\imath/\kappa)(\, j_l(\kappa) + \kappa j_l'(\kappa)\,)(\, h_l^{(1)}(\kappa) + \kappa h_l^{(1)\prime}(\kappa)\,)$$

$$\langle \text{EFIE}_\kappa(\mathbf{X}_{l,m}^-), \mathbf{X}_{l,m}^-\rangle = (\imath/\kappa)j_l(\kappa)h_l^{(1)}(\kappa)$$

$$\langle \text{MFIE}_\kappa(\mathbf{X}_{l,m}^+), \mathbf{X}_{l,m}^-\rangle = \langle \text{MFIE}_\kappa(\mathbf{X}_{l,m}^-), \mathbf{X}_{l,m}^+\rangle$$
$$= -\imath[j_l(\kappa)(h_l^{(1)}(\kappa) + \kappa h_l^{(1)\prime}(\kappa)) + h_l^{(1)}(\kappa)(j_l(\kappa) + \kappa j_l'(\kappa))]$$

$$\langle \text{EFIE}_\kappa(\mathbf{X}_{l,m}^+), \mathbf{X}_{l,m}^-\rangle = \langle \text{EFIE}_\kappa(\mathbf{X}_{l,m}^-), \mathbf{X}_{l,m}^+\rangle = 0$$

$$\langle \text{MFIE}_\kappa(\mathbf{X}_{l,m}^+), \mathbf{X}_{l,m}^+\rangle = \langle \text{MFIE}_\kappa(\mathbf{X}_{l,m}^-), \mathbf{X}_{l,m}^-\rangle = 0$$

References

[1] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX).